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Some Remarks on Turán's Inequality

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We examine how large the L^p norm on $[-1, 1]$ of the derivative of a real algebraic polynomial of degree at most n with zeros only in $[-1, 1]$ can be if the L^q norm of the polynomial is 1.

Let H_n be the class of real algebraic polynomials of degree n, whose zeros all lie in the interval $[-1, 1]$. Let R_n be the class of real trigonometric polynomials of degree n with only real roots. Let

$$\|f\|_{C[a,b]} = \|f\|_{L^\infty[a,b]} = \max_{a \leqslant x \leqslant b} |f(x)|,$$

$$\|f\|_{L^p[a,b]} = \left(\int_a^b |f(x)|^p \, dx\right)^{1/p} \qquad \text{for} \quad 1 \leqslant p < \infty.$$

In 1939, P. Turán [4] proved that for $f \in H_n$,

$$\|f'\|_{C[-1,1]} \geqslant Cn^{1/2} \|f\|_{C[-1,1]},$$

where here and throughout the whole paper, C always indicates some positive absolute constant. Later, J. Erőd [3] improved the constant above. In 1976, A. K. Varma [5] established the corresponding inequality in L^2 space. The general form for $1 \leqslant p \leqslant \infty$ of the inequality is due to the author [7]:

THEOREM 1. *If $f \in H_n$, then for $1 \leqslant p \leqslant \infty$,*

$$\|f'\|_{L^p[-1,1]} \geqslant Cn^{1/2} \|f\|_{L^p[-1,1]}.$$

In 1983, A. K. Varma [6] obtained the best possible constant in this inequality when $p = 2$. In 1986, V. F. Babenko and S. A. Pichugov [1, 2]

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found the best possible constants in this inequality and in the corresponding trigonometric case when $p=\infty$.

In [8] we considered similar results in L^p for $0<p<1$.

The object of this paper is to establish the following more general result.

THEOREM 2. *If* $f\in H_n$, *then for* $1\leqslant p\leqslant q\leqslant\infty$,

$$\|f'\|_{L^p[-1,1]}\geqslant Cn^{1/2-1/(2p)+1/(2q)}\|f\|_{L^q[-1,1]}. \qquad (*)$$

Proof. Denote by $-1\leqslant x_1<x_2<\cdots<x_k\leqslant 1$ all the distinct zeros of $f\in H_n$, and the order of x_i by m_i. Write the maximum point of $|f(x)|$ between (x_i, x_{i+1}) as α_i. If $x_1>-1$ or $x_k<1$, set $\alpha_0=-1$ or $\alpha_k=1$. Also, let

$$m(x)=\sum_{i=1}^{k}\frac{m_i}{x-x_i}.$$

We divide the proof into several cases.

Case 1. $\alpha_j-x_j\leqslant n^{-1/2}$ for some j. Then applying Hölder's inequality we have

$$\int_{x_j}^{\alpha_j}|f'(x)|^p\,dx\geqslant n^{(p-1)/2}|f(\alpha_j)|^p\geqslant n^{(p-1)/2}(\alpha_j-x_j)^{-p/q}\left(\int_{x_j}^{\alpha_j}|f(x)|^q\,dx\right)^{p/q},$$

that is,

$$\int_{x_j}^{\alpha_j}|f'(x)|^p\,dx\geqslant n^{(pq-q+p)/(2q)}\left(\int_{x_j}^{\alpha_j}|f(x)|^q\,dx\right)^{p/q}.$$

If $\alpha_j-x_j\geqslant n^{-1/2}$ for some j, put $d_j=[(\alpha_j-x_j)n^{1/2}]$, where $[x]$ denotes the greatest integer not exceeding x.

Case 2.1. $\alpha_j-x_j>n^{-1/2}$ for some j. Then as above,

$$\int_{x_j}^{\alpha_j-d_j/n^{1/2}}|f'(x)|^p\,dx\geqslant n^{(pq-q+p)/(2q)}\left(\int_{x_j}^{\alpha_j-d_j/n^{1/2}}|f(x)|^q\,dx\right)^{p/q}.$$

Case 2.2. $\alpha_j-x_j>n^{-1/2}$ for some j, and $|f(\alpha_j-tn^{-1/2})|\geqslant 2|f(\alpha_j-(t+1)n^{-1/2})|$ for some $t=0, 1, ..., d_j-1$. Using Hölder's inequality again, we have

$$\int_{\alpha_j-(t+1)/n^{1/2}}^{\alpha_j-t/n^{1/2}}|f'(x)|^p\,dx\geqslant n^{(p-1)/2}|f(\alpha_j-tn^{-1/2})-f(\alpha_j-(t+1)n^{-1/2})|^p,$$

and since

$$|f(\alpha_j - tn^{-1/2})| \geqslant 2\, |f(\alpha_j - (t+1)n^{-1/2})|,$$

we get

$$\int_{\alpha_j-(t+1)/n^{1/2}}^{\alpha_j - t/n^{1/2}} |f'(x)|^p\, dx \geqslant 2^{-p} n^{(pq-q+p)/(2q)} \left(\int_{\alpha_j-(t+1)/n^{1/2}}^{\alpha_j - t/n^{1/2}} |f(x)|^q\, dx \right)^{p/q}.$$

Case 2.3. $\alpha_j - x_j > n^{-1/2}$ for some j, and $|f(\alpha_j - rn^{-1/2})| < 2\,|f(\alpha_j - (r+1)n^{-1/2})|$ for some $r = 0, 1, ..., d_j - 1$. Applying the mean value theorem to the function $f'(x)/f(x)$ on the interval $(x_j, \alpha_j]$ and noting that $f'(\alpha_j) = 0$, we have for $x \in (x_j, \alpha_j - \frac{1}{2}n^{-1/2})$, $j \neq k$,

$$m(x) = \sum_{i=1}^{k} \frac{m_i}{x - x_i} = (\alpha_j - x) \sum_{i=1}^{k} \frac{m_i}{(\xi - x_i)^2} \geqslant \frac{n^{1/2}}{8}, \qquad \xi \in (x, \alpha_j);$$

a similar calculation leads to

$$m(x) \geqslant \sum_{i=1}^{k} \frac{m_i}{x - x_i} - \sum_{i=1}^{k} \frac{m_i}{\alpha_k - x_i} \geqslant \frac{n^{1/2}}{8}$$

for $x \in (x_k, \alpha_k - \frac{1}{2}n^{-1/2})$. So

$$\begin{aligned} \int_{\alpha_j-(r+1)/n^{1/2}}^{\alpha_j - r/n^{1/2}} |f'(x)|^p\, dx &\geqslant \int_{\alpha_j-(r+1)/n^{1/2}}^{\alpha_j - (2r+1)/(2n^{1/2})} |f(x)|^p\, |m(x)|^p\, dx \\ &\geqslant 2^{-3p-1} n^{(p-1)/2}\, |f(\alpha_j - (r+1)n^{-1/2})|^p \\ &\geqslant 2^{-4p-1} n^{(p-1)/2}\, |f(\alpha_j - rn^{-1/2})|^p; \end{aligned}$$

hence

$$\int_{\alpha_j-(r+1)/n^{1/2}}^{\alpha_j - r/n^{1/2}} |f'(x)|^p\, dx \geqslant 2^{-4p-1} n^{(pq-q+p)/(2q)} \left(\int_{\alpha_j-(r+1)/n^{1/2}}^{\alpha_j - r/n^{1/2}} |f(x)|^q\, dx \right)^{p/q}.$$

Now Cases 2.1–2.3 prove that

$$\int_{x_j}^{\alpha_j} |f'(x)|^p\, dx \geqslant 2^{-4p-1} n^{(pq-q+p)/(2q)} \left(\int_{x_j}^{\alpha_j} |f(x)|^q\, dx \right)^{p/q}$$

for $\alpha_j - x_j > n^{-1/2}$, while Case 1 proves the inequality if $\alpha_j - x_j \leqslant n^{-1/2}$. The same inequality clearly holds if $\int_{x_j}^{\alpha_j}$ is replaced with $\int_{\alpha_j}^{x_{j+1}}$. Summing over all j, we get the claimed inequality. ∎

With $f(x) = (1 - x^2)^{[n/2]}$, we see that the order $n^{1/2 - 1/(2p) + 1/(2q)}$ in (∗) cannot be improved.

The proofs for the following theorems are similar.

THEOREM 3. *Let* $1 \leqslant p \leqslant q \leqslant \infty$, *and* $f(x)$ *be an* n*th degree algebraic polynomial, which has only real roots. If at most* k *roots of* $f(x)$ *lie outside the interval* $[-1, 1]$, *then*

$$\|f'\|_{L^p[-1,1]} \geqslant C_k n^{1/2-1/(2p)+1/(2q)} \|f\|_{L^q[-1,1]},$$

where C_k *is a positive constant depending only upon* k.

THEOREM 4. *Let* $1 \leqslant p \leqslant q \leqslant \infty$. *If* $f \in R_n$, *then*

$$\|f'\|_{L^p[0,2\pi]} \geqslant C n^{1/2-1/(2p)+1/(2q)} \|f\|_{L^q[0,2\pi]}.$$

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